Fast and Stable Numerical Method for Boundary-Layer Flow with Massive Blowing

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THE structure of a laminar boundary layer with massive blowing is greatly complicated in view of the viscid-inviscid interactions. These interactions result in larger deflection of incident flow and also a significant alteration in the major flow variables including pressure distribution. To date various analytical models[1] have been developed for the prediction of the flowfield. More recently, numerical studies based on the laminar boundary-layer approximation have been developed for the prediction of a boundary-layer flow structure with a prescribed pressure gradient. Methods adopted in the numerical investigation include backward shooting, Libby,[2] Nachtsheim and Green,[3] Liu and Nachtscheim,[4] and matrix method Garrett, Smith, and Perkins.[5]

Although it has been shown by Liu and Nachtsheim[6] that the backward shooting technique is stable, its computing time is long and increases as injection rates increase. Therefore, from an economic point of view, it is highly desirable to devise another stable and yet fast numerical scheme. With this motivation, the authors tried the matrix method resulting from various finite differencing. However, for differential equations with large parasitic eigenvalues such as massive blowing problem, most differencing schemes result in a matrix with large condition number and therefore is unstable. It is the purpose of the present study to propose a fast, stable, and easy to code matrix method of which the number of nodes and rate of convergence (and therefore the computing time) are independent of injection rate.

I. Method Description

When differential equations are well behaved, it is common practice to use central difference scheme with equal spacing because it is more accurate. However, when differential equations are unstable as in the present case, it is necessary to use noncentered or unequal spacing difference scheme. The windward differencing, which has been successfully employed by Ref. 5, is noncentered. As will be shown in Sec. II, the unequal spacing difference scheme proposed in Sec. I result in an irreducible matrix with direction graph strongly connected.[6]

Consider the Cohen-Peshotko equations

$$f''' + ff'' + 0.5(1 + S - f'^2) = 0 \tag{1}$$

$$S'' + fs' = 0 \tag{2}$$

$$f(0) = f_w,\ f'(0) = 0,\ S(0) = S_w \tag{3}$$

$$f'(\infty) = 1,\ S(\infty) = 0 \tag{4}$$

This is the same set of equations considered in Refs. 3 and 4. For negative f_w, the stream function f is negative near the wall and positive at the edge and there is a dividing streamline where $f(\eta_0) = 0$. For $(-f_w) >> 1$, the location of the dividing streamline increases with $(-f_w)$ and becomes very large. As has been shown by Liu and Nachtscheim,[6] the parasitic eigenvalues of the equations in the inner region, $\eta > \eta_0$, is proportional to $(-f)$ and is, therefore, very unstable.

We introduce a new dependent variable u, and rewrite Eqs. (1-4) as follows

$$f' = u + 1 \tag{5}$$

$$u'' + fu' - 0.5(S - 2u - u^2) = 0 \tag{6}$$

$$S'' + fS' = 0 \tag{7}$$

$$f(0) = f_w,\ u(0) = -1,\ S(0) = S_w \tag{8}$$

$$u' + \left(f + \frac{2\beta + 1}{f}\right)u = 0,\ (1 - 1/f^2)S' + fS = 0 \tag{9}$$

where the asymptotic relations for large η have been used for large but finite η.[7] Applying the quasilinearlization to Eqs. (5-7), we obtain

$$f' - u = 0 \tag{10}$$

$$u'' + {}^{(k)}fu' - ({}^{(k)}u + 1)u + {}^{(k)}u'f + 0.5S = {}^{(k)}f^{(k)}u' - 0.5^{(k)}u^2 \tag{11}$$

$$S'' + {}^{(k)}fS' + {}^{(k)} + {}^{(k)}S_f' = {}^{(k)}f^{(k)}S' \tag{12}$$

where the left superscripts denote functions evaluated at kth iteration. The two-point boundary value problem of Eqs. (10-12) is now linear in the variables of $(k+1)$th iteration. Many schemes have been devised for solving linear boundary value problem of Eqs. (10-12) type. However, if the boundary conditions are such that Eqs. (10-12) are ill-behaved, as in the present case, most of the existing schemes fail. The procedure we shall present is stable and very efficient.

We place an arbitrary set of nodes of $0 < \eta < \eta_\infty$ and use the notation

$$\eta_1 = 0,\ \eta_{j+1} = \eta_j + h_j,\ j = 1,2,\ldots,J,\ \eta_{J+1} = \eta_\infty \tag{13}$$

The difference approximations to Eqs. (10-12) are defined, for $2 < j < J$, by

$$\alpha_{j-1}f_{j-1} + \alpha_j f_j + \alpha_{j+1}f_{j+1} - u_j = 1 \tag{14}$$

$$\gamma_{j-1}u_{j-1} + \gamma_j u_j + \gamma_{j+1}u_{j+1} + {}^{(k)}f_j(\alpha_{j-1}u_{j-1} + \alpha_j u_j + \alpha_{j+1}u_{j+1}) - ({}^{(k)}u_j + 1)u_j + {}^{(k)}u_j' f_j + 0.5S_j = {}^{(k)}f_j{}^{(k)}u_j' - 0.5^{(k)}u_j^2 \tag{15}$$

$$\gamma_{j-1}S_{j-1} + \gamma_j S_j + \gamma_{j+1}S_{j+1} + {}^{(k)}f_j(\alpha_{j-1}S_{j-1} + \alpha_j S_j + \alpha_{j+1}S_{j+1}) + {}^{(k)}S_j' f_j = {}^{(k)}f_j{}^{(k)}S_j' \tag{16}$$

where

$$\alpha_{j-1} = \frac{-h_j}{h_{j-1}(h_j + h_{j-1})},\ \alpha_j = \frac{h_j - h_{j-1}}{h_{j-1}h_j},\ \alpha_{j+1} = \frac{h_{j-1}}{h_j(h_j + h_{j-1})} \tag{17}$$

$$\gamma_{j-1} = \frac{2}{h_{j-1}(h_j + h_{j-1})},\ \gamma_j = \frac{-2}{h_{j-1}h_j},\ \gamma_{j+1} = \frac{2}{h_j(h_j + h_{j-1})} \tag{18}$$

Received April 7, 1975; revision received August 25, 1975.
Index category: Boundary Layers and Convective Heat Transfer—Laminar.

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The boundary conditions, Eq. (8-9), simply become

$$f_1 = f_w,\ u_1 = -1,\ S_1 = S_w \tag{19}$$

$$(u_{J+1} - u_J)/h_J + ({}^{(k)}f_{J+1} + 2/{}^{(k)}f_{J+1})u_{j=1} = 0 \tag{20}$$

$$(1 - 1/{}^{(k)}f^2_{J+1})(S_{J+1} - S_J)/h_J + {}^{(k)}f_{J+1}S_{J+1} = 0$$

After rearrangement, the linear system of Eqs. (14-16, 19, 20) can be written in block-tridiagonal matrix form as

$$\begin{array}{cccccccc} A_1 & B_1 & 0 & 0 & . & . & Y_1 & D_1 \\ C_2 & A_2 & B_2 & 0 & . & . & Y_2 & D_2 \\ 0 & C_3 & A_3 & B_3 & . & . & Y_3 & = D_3 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & C_J & A_J & B_J & Y_J & D_J \\ . & . & . & 0 & C_{J+1} & A_{J+1} & Y_{J+1} & D_{J+1} \end{array} \tag{21}$$

II. Stability Analysis

The system of equations, Eq. (21), can be solved by a more or less standard block-tridiagonal factorization procedure.[9] For low blowing, the vector-matrix Eq. (21) is block diagonally dominant with respect to the matrix norm ‖ ‖. However, as blowing increases, the condition number (10-11) of the matrix increases and the factorization procedure is unstable if equal spacing, i.e. if $h_j = h_{j+1}$ for $j = 1,...,J$ is used. To restore the block diagonal dominance, a proper choice of the stepsize h_j is important. To prove this point, let us examine the norms of matrices C_J, A_j, B_j.

$$\|C_j\| = |\gamma_{j-1} + \alpha_{j-1}{}^{(k)}f_j| \tag{22}$$

$$\|A_j\| \cdot {}^2 = \text{maximum eigenvalue of the matrix } A_jA_j^T \tag{23}$$

$$\|B_j\| = |\gamma_{j+1} + \alpha_{j+1}{}^{(k)}f_j| \tag{24}$$

Recalling that for massive blowing we have[12]

$$u' = 0.5(1+S_w)/(1-f_w) = \tau << 1 \tag{25}$$

$$u = \tau\eta \tag{26}$$

$$f = f_w + \tau\eta^2/2 \tag{27}$$

and S' is negligible in the inner layer, and thus

$$\|C_j\| + \|B_j\| = |\gamma_{j-1} + \alpha_{j-1}{}^{(k)}f_j| + |\gamma_{j+1} + \alpha_{j+1}{}^{(k)}f_j| \tag{28}$$

$$\|A_j\| \approx |\gamma_j| \tag{29}$$

if small and constant stepsize is used. Therefore,

$$\|C_j\| + \|B_j\| > \|A_j\| \tag{30}$$

i.e. the block LU decomposition is numerically unstable. Indeed, the numerical experiment performed with constant stepsize, $h_j = 0.4$, for $f_w < -4$ is not only divergent but also oscillatory. However, if stepsize h_j is chosen such that

$$\alpha_j{}^{(k)}f_j >> \gamma_j \tag{31}$$

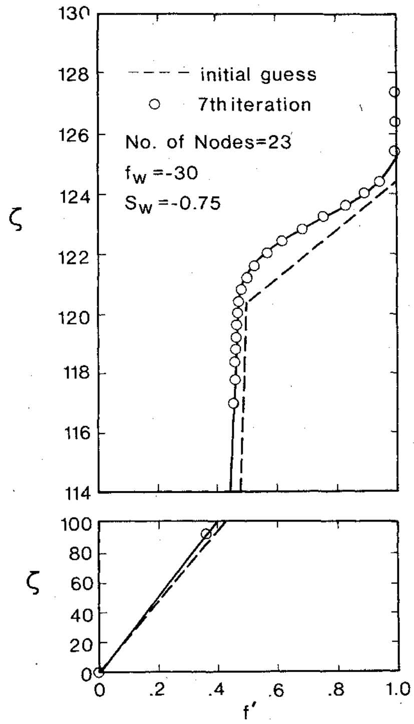

Fig. 1 Velocity profile.

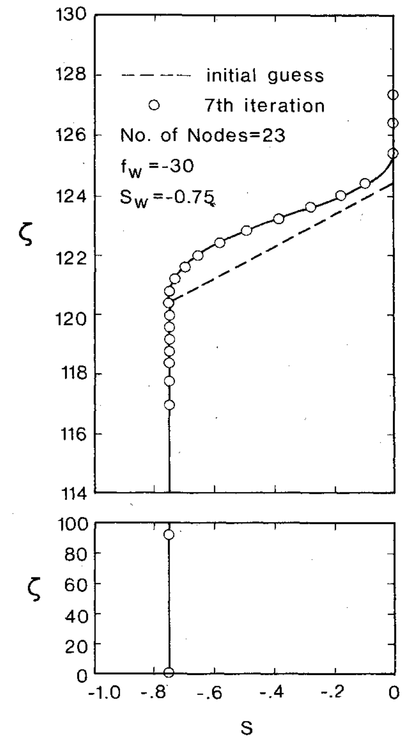

Fig. 2 Enthalpy profile.

i.e. unequal stepsize with very large stepsize near the wall and decreasing toward the dividing streamline, then

$$\frac{\|A_j\|}{\|C_j\| + \|B_j\|} \approx \frac{|\alpha_j|}{|\alpha_{j-1}| + |\alpha_{j+1}|} \geqslant 1 \tag{32}$$

and is block diagonally dominant. Therefore, the block LU decomposition is numerically stable and, in fact[13]

$$\|L_j\| < \|A_j\|/\|B_j\|,\ \|U_j\| < \|B_j\| + \|A_j\| \tag{33}$$

We note that in the inner layer, the flow is almost inviscid for massive blowing and there is no danger of losing accuracy by choosing a big stepsize as recommended by Eq. (31).

III. Results and Discussion

To the authors' knowledge, there exist at least two other approaches to solve boundary-layer equations resulting in block tri-diagonal matrix. One is that by Keller and Cebeci,[14-16] and the other is that by Libby and Wu.[17-18] Liu and Davy have tried them both.[11] Both approaches give accurate results for low blowing. However, neither of them prove to be successful for $f_w < -4$ even with very good initial guesses. In an unpublished note, the first author and M.J. Green of Ames Research Center calculate the condition numbers resulting from both matrices. We are able to prove that the condition numbers increase with $|f_w|$ and therefore the factorization procedure is unstable.

The numerical method we have discussed in the previous sections was coded in FORTRAN and calculations have beem made for the case of very large blowing $f_w = -30$. The results obtained from IBM 360-67 computer are shown in Figs. 1 and 2. The initial guesses are shown by the dashed lines. The inner portion of the initial guess is that of Libby's zeroth order inner solution.[12] The node points are chosen so that Eq. (31) is satisfied. The convergence is reached in 7th iteration with an error less than 1% for all the node points. Each iteration takes about 2 sec. Several cases with different β, f_w, and S_w have been tried. They all converged in less than 10 iterations and over all computing time is little and independent of blowing parameter.

The same scheme has been tried on a non-similar turbulent boundary layer with mass and heat transfer over a rough surface with two-equation model for closure. No difficulty has been encountered yet. We shall report the results elsewhere.

References

[1] Wallace, J. and Kemp, N., "Similarity Solutions to the Massive Blowing Problem," *AIAA Journal,* Vol. 7, Aug. 1969, pp. 1517-1523.

[2] Libby, P.A., "The Homogeneous Boundary Layer at an Axisymmetric Stagnation Point with Large Rate of Injection," *Journal of the Aerospace Sciences,* Vol. 29, No. 1, Jan. 1962, pp. 48-60.

[3] Nachtsheim, P.R. and Green, M.J., "Numerical Solution of Boundary Layers with Massive Blowing," *AIAA Journal,* Vol. 9, March 1970, pp. 533-535.

[4] Liu, T.M. and Nachtsheim, P.R., "Shooting Method for Solution of Boundary Layer Flows with Massive Blowing," *AIAA Journal,* Vol. 11, Nov. 1973, pp. 1584-1586.

[5] Garrett, L.B., Smith, G.L. and Perkins, J.N., "An Implicit Finite-Difference Solution fo the Viscous Shock Layer, Including the Effects of Radiation and Strong Blowing," NASA TR R-388, Nov. 1972.

[6] Liu, T.M. and Nachtsheim, P.R., "Numerical Stability of Boundary Layers with Massive Blowing," *AIAA Journal,* Vol. 11, Aug. 1973, pp. 1197-1198.

[7] Varga, R.S., *Matrix Iterative Analysis,* Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 19-22.

[8] Libby, P.A. and Liu, T.M., "Some Similar Laminar Flows Obtained by Quasi-linearization," *AIAA Journal,* Vol. 6, Aug. 1968, pp. 1541-1548.

[9] Isaacson, E. and Keller, H.B., *Analysis of Numerical Methods,* Wiley, N.Y., 1966, pp. 58-61.

[10] Wilkinson, J.H., *The Algebraic Eigenvalue Problem,* Clarendon Press, Oxford, 1965, p. 91.

[11] Liu, T.M. and Davy, W.C., "Nonequilibrium Boundary Layer at a Stagnation Point for a Hydrogen-Helium Stream over Ablating Graphite," *Acta Astronautica,* Vol. 1, No. 4, April 1974, pp. 485-503.

[12] Libby, P.A., "On the Numerical Analysis of Stagnation Point Flows with Massive Blowing," *AIAA Journal,* Vol. 8, Nov. 1970, pp. 2095-2096.

[13] Varah, J.M., "On the Solution of Block-Tridiagonal Systems Arising from Certain Finite-Difference Equations," *Mathematics of Computation,* Vol. 26, No. 20, Oct. 1972, pp. 859-868.

[14] Keller, H.B. and Cebeci, T., "Accurate Numerical Methods for Boundary Layer Flows, I. Two-Dimensional Laminar Flows," *Lecture Notes in Physics, Proceedings of Second International Conference on Numerical Methods in Fluid Dynamics,* Springer-Verlag, N.Y., 1971.

[15] Keller, H.B. and Cebeci, T., "Accurate Numerical Methods for Boundary Layer Flows,II. Two-Dimensional Turbulent Flow," *AIAA Paper,* 71-164, New York, N.Y., 1971.

[16] Keller, H.B. and Cebeci, T., "An Inverse Problem in Boundary-Layer Flows: Numerical Determination of Pressure Gradient for a Given Wall Shear," *Journal of Computational Physics,* Vol. 10, 1972, pp. 151-161.

[17] Wu, P. and Libby, P.A., "Further Results on the Stagnation Point Boundary Layer," *Combustion Sciences and Technology,* Vol. 6, No. 3, 1972, pp. 159-168.

[18] Wu, P. and Libby, P.A., "Non-Similar Flows Between the Solution Branches of the Falkner-Skan Equation," *AIAA Journal,* Vol. 11, Jan. 1973, pp. 112-113.

Stresses and Displacements in Rotating Anisotropic Disks with Variable Densities

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THE problem of rotating disks was first treated in the early nineteenth century. Solutions of the isotropic disks, including variable thickness, variable density, and other cases, can be found in most of the standard elasticity textbooks. Therefore, rotating disks were considered to be one of the exhausted subjects in the field of solid mechanics. Recently, new interest has been generated to reinvestigate this centuries-old problem. This new interest is the result of the development of composite materials and their applications.

Composite materials are characterized by high strength-to-weight ratios, heterogeneity and anisotropic characteristics. Because of the heterogeneity and anisotropic characteristics, it makes the effort in predicting the material response under multiaxial states of stress more difficult.[1] Recently, the rotating disk technique has been proved to be a simple and reliable means of generating a biaxial state of stress when the loads cannot be directly applied to the material under investigation. In this case, an analytical solution of this problem is needed to interpret the experimental results generated in the laboratory. Recently, because of the steady increase in the use of energy and the impact of that use on the environment, a flywheel made of composite materials has been proved to be an efficient means of storing energy.[2,3] Likewise, analytical solution of rotating disks made of composite materials is essential in design and analysis.

Tang[4] and Murthy and Sherbourne[5] have treated the cases of rotating polar orthotropic disks of uniform and nonuniform thickness, respectively. Reddy and Srinath[6] further investigated the effect of material density on the stresses and displacements of a rotating polar orthotropic circular disk. In recent investigations, closed-form solutions of rotating circular and elliptical disks made of general orthotropic materials have been developed by the author.[7,8]

Density variation in the material is, however, inevitable, and it has been shown that density variation has significant effects on the stresses and displacements of a rotating polar orthotropic disk. It is the purpose of this note to present a closed-form solution of a rotating orthotropic circular disk of which a radial density variation is assumed. A semi-inverse technique is employed, and the material is assumed to be linear elastic.

Analysis

For an orthotropic cylinder, if the axial displacement vanishes and the displacement components on any plane per-

Received April 28, 1975; revision received July 14, 1975.

Index categories: Structural Composite Materials (including Coatings); Structural Static Analysis.

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